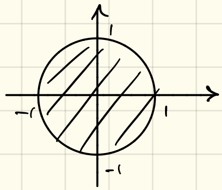
Série 5

Ex 1 : 1) $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$



coordonnées polaires

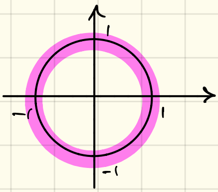
$x = r\cos\theta, \quad y = r\sin\theta. \qquad r \geq 0, \quad \theta \in [0, 2\pi]$

$x^2 + y^2 < 1 \iff r^2 < 1$

$\overset{r \geq 0}{\iff} 0 \leq r < 1$

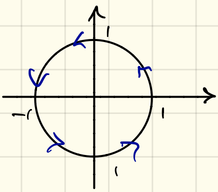
Paramétrisation de A : $x = r\cos\theta$, $y = r\sin\theta$, $r \in [0,1]$, $\theta \in [0, 2\pi]$.

∂A :



Le bord de A est le cercle de rayon 1 :

$\gamma(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$.



orientation positive ✓

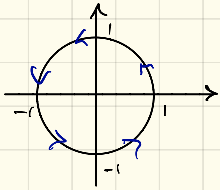
technique 2 : transformer les inégalités dans la description du A en égalités

$$\partial A = \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}.$$

coordonnées polaires

$x = r\cos\theta$, $y = r\sin\theta$. $\quad r \geq 0$, $\theta \in [0, 2\pi]$

$x^2+y^2 = 1 \iff r = 1 \quad \Rightarrow \gamma(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$

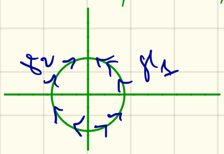


orientation positive ✓

Remarque :

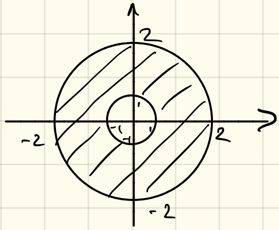
On aurait aussi pu résoudre pour x : $x = \pm\sqrt{1-y^2} \quad y \in [-1, 1]$

$$\Rightarrow \quad \gamma_1(t) = \left(\sqrt{1-t^2}, t\right), \quad \gamma_2(t) = \left(-\sqrt{1-t^2}, t\right)$$



2) $A = \{(x,y) \in \mathbb{R}^2 : 1 < x^2+y^2 < 4\}$

coordonnées polaires



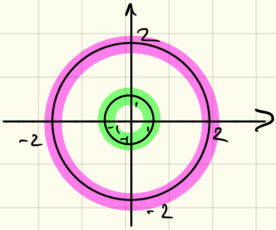
$x = r\cos\theta,\ y = r\sin\theta$, $r \geq 0$, $\theta \in [0, 2\pi]$

$1 < x^2+y^2 < 4 \iff 1 < r^2 < 4$

$\overset{r \geq 0}{\iff} 1 < r < 2$

Paramétrisation de A : $x = r\cos\theta$, $y = r\sin\theta$, $r \in [1,2]$, $\theta \in [0, 2\pi]$

∂A :

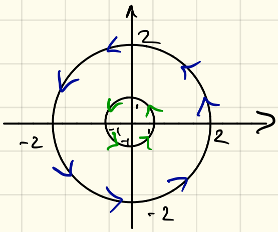


Le bord de A a deux parties :

- Le cercle de rayon 1
- Le cercle de rayon 2

$\gamma_1(t) = (\cos t, \sin t)$ $t \in [0, 2\pi]$

$\gamma_2(t) = (2\cos t, 2\sin t)$



γ_1 est orienté négativement

γ_2 est orienté positivement.

Ex2 1)

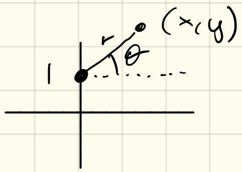
$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 < 1\}$$

On reconnaît l'équation d'un disque centré en $(0,1)$ (Rappel : l'équation d'un disque centré en (x_0, y_0) de rayon R est

$$(x-x_0)^2 + (y-y_0)^2 < r^2)$$

On passe en coordonnées polaires centrées en $(0,1)$:

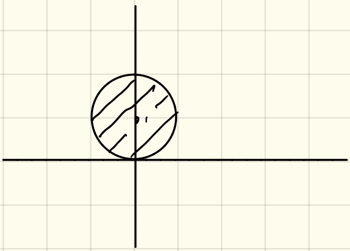
$$x = r\cos\theta, \quad y = 1 + r\sin\theta \qquad r \geq 0, \ \theta \in [0, 2\pi]$$



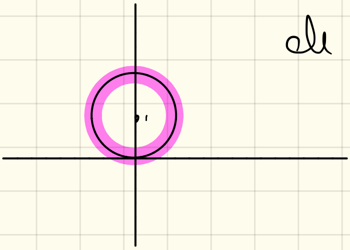
$$x^2 + (y-1)^2 < 1 \iff r^2 < 1$$

$$\overset{r \geq 0}{\iff} \quad 0 \leq r \leq 1$$

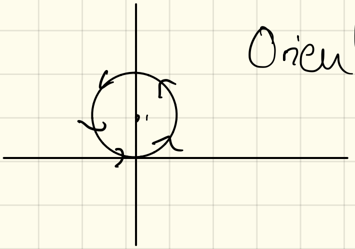
Paramétrisation de A : $x = r\cos\theta$, $y = 1 + r\sin\theta$, $r \in [0,1]$, $\theta \in [0, 2\pi]$



∂A : Le bord de A est le cercle centré en $(0,1)$ de rayon 1 :



$$\gamma(t) = (\cos t, 1+\sin t)$$
$$t \in [0, 2\pi]$$



Orienté positivement.

Détails de calcul :

$$\iint_A \operatorname{rot} F \,dx\,dy = \int_0^1 \int_0^{2\pi} \left((r\cos\theta)^2 + (1 - r\sin\theta)^2 \right) r \,d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} \left(r^2\cos^2\theta + 1 - 2r\sin\theta + r^2\sin^2\theta \right) r \,d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} r + \underbrace{r^3\cos^2\theta - 2r^2\sin\theta + r^3\sin^2\theta}_{= r^3} \,d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} r + r^3 - 2r^2\sin\theta \,d\theta\,dr = 2\pi \left[\frac{1}{2} r^2 + \frac{1}{4} r^4 \right]_0^1$$

$$= \frac{3\pi}{2}$$

$$\int_{\partial A} F \cdot dl = \int_0^{2\pi} \left\langle \left(-(1+\sin\theta)\cos^2\theta,\ (1+\sin\theta)^2\cos\theta \right), (-\sin\theta, \cos\theta) \right\rangle d\theta$$

$$= \int_0^{2\pi} \sin\theta\cos^2\theta + \sin^2\theta \underbrace{\cos^2\theta}_{=1-\sin^2\theta} + \cos^2\theta + 2\cos^2\theta\sin\theta + \sin^2\theta\underbrace{\cos^2\theta}_{=1-\sin^2\theta} d\theta$$

$$= \int_0^{2\pi} 3\cos^2\theta\sin\theta + 2\sin^2\theta - 2\sin^4\theta + \cos^2\theta \, d\theta$$

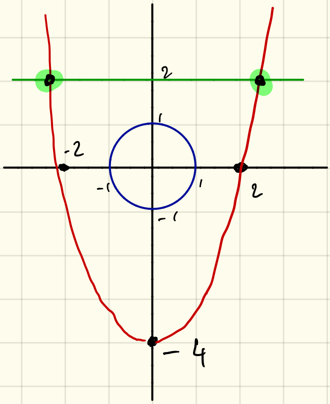
$$= \sin^2\theta + 1$$

$$= \int_0^{2\pi} 1 + \sin^2\theta + 3\cos^2\theta\sin\theta - 2\sin^4\theta \, d\theta$$

$$= \left[\theta + \frac{1}{2}\left[\theta - \cos\theta\cancel{\sin\theta}\right] - \cancel{\cos^3\theta} - 2\left[\frac{3}{8}\theta - \frac{1}{4}\cancel{\sin(2\theta)} + \frac{1}{32}\cancel{\sin(4\theta)}\right] \right]_0^{2\pi}$$

$$= 2\pi + \pi - \frac{3}{2}\pi = \frac{3\pi}{2}$$

2)



$x^2+y^2=1$

$x^2-4=y$

$y=2$

$\begin{cases} x^2-4=y \\ y=2 \end{cases} \Rightarrow x^2-4=2 \Rightarrow x^2=6 \Rightarrow x=\pm\sqrt{6}$

- Si $A_0=\{(x,y)\in\mathbb{R}^2 : x^2-4\leq y\leq 2\}$ & $A_1=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$, alors, $A=A_0\setminus A_1$ et

$$\iint_A \operatorname{rot} F(x,y)\,dx\,dy = \iint_{A_0} \operatorname{rot} F(x,y)\,dx\,dy - \iint_{A_1} \operatorname{rot} F(x,y)\,dx\,dy$$

Paramétrisation de $A_0=\{(x,y)\in\mathbb{R}^2 : x^2-4\leq y\leq 2\}$

A_0 est y simple :

$$\iint_{A_0} \operatorname{rot} F(x,y)\,dx\,dy = \int_{-\sqrt{6}}^{\sqrt{6}}\int_{x^2-4}^{2} \operatorname{rot} F(x,y)\,dy\,dx.$$

(A_0 est aussi x-simple :

$$\iint_{A_0} \operatorname{rot} F(x,y)\,dx\,dy = \int_{-4}^{2}\int_{-\sqrt{4+y}}^{\sqrt{4+y}} \operatorname{rot} F(x,y)\,dx\,dy$$)

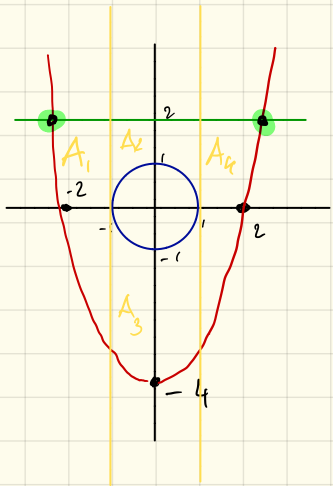
Paramétrisation de $A_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$

coordonnées polaires

$x = r\cos\theta, \quad y = r\sin\theta, \qquad r\in[0,1], \quad \theta\in[0,2\pi]$

(voir ex 1, 1))

- Alternativement, on peut séparer A en 4 domaines y simples :



$$A_1 = \{(x,y)\in\mathbb{R}^2 : x\in[-\sqrt{6},-1],\ x^2-4\le y\le 2\},$$

$$A_2 = \{(x,y)\in\mathbb{R}^2 : x\in[-1,1],\ \sqrt{1-x^2}\le y\le 2\}$$

$$A_3 = \{(x,y)\in\mathbb{R}^2 : x\in[-1,1],\ x^2-4\le y\le -\sqrt{1-x^2}\}$$

$$A_4 = \{(x,y)\in\mathbb{R}^2 : x\in[1,\sqrt{6}],\ x^2-4\le y\le 2\}.$$

On a $A = A_1\cup A_2\cup A_3\cup A_4$ et donc,

$$\iint_A \operatorname{rot} F(x,y)\,dx\,dy = \iint_{A_1} \operatorname{rot} F(x,y)\,dx\,dy + \iint_{A_2} \operatorname{rot} F(x,y)\,dx\,dy$$
$$+ \iint_{A_3} \operatorname{rot} F(x,y)\,dx\,dy + \iint_{A_4} \operatorname{rot} F(x,y)\,dx\,dy$$

$$\iint_{A_1} \operatorname{rot} F(x,y)\,dx\,dy = \int_{-\sqrt{6}}^{-1} \int_{x^2-4}^{2} -x \,dy\,dx$$

$$= \int_{-\sqrt{6}}^{-1} \Big[-xy\Big]_{y=x^2-4}^{y=2} dy = \int_{-\sqrt{6}}^{-1} -2x + x^3 - 4x \,dx$$

$$= \int_{-\sqrt{6}}^{-1} x^3 - 6x \,dx = \Big[\tfrac{1}{4}x^4 - 3x^2\Big]_{-\sqrt{6}}^{-1} = \frac{1}{4} - 3 - \frac{36}{4} + 18$$

$$= \frac{1}{4} + 15 - 9 = 6 + \frac{1}{4}$$

$$\iint_{A_2} \operatorname{rot} F(x,y)\,dx\,dy = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{2} -x\,dy\,dx = \int_{-1}^{1} \Big[-xy\Big]_{y=\sqrt{1-x^2}}^{y=2} dx$$

$$= \int_{-1}^{1} x\sqrt{1-x^2} - 2x \,dx = \Big[-\tfrac{1}{3}(1-x^2)^{3/2} - x^2\Big]_{x=-1}^{x=1}$$

$$= -0 - 1 + 0 + 1 = 0.$$

$$\iint_{A_3} \operatorname{rot} F(x,y)\,dx\,dy = \int_{-1}^{1} \int_{x^2-4}^{-\sqrt{1-x^2}} -x\,dy\,dx = \int_{-1}^{1} \Big[-xy\Big]_{y=x^2-4}^{y=-\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} x^3 - 4x + x\sqrt{1-x^2}\,dx = \Big[\tfrac{1}{4}x^4 - 2x^2 - \tfrac{1}{3}(1-x^2)^{3/2}\Big]_{x=-1}^{x=1}$$

$$= \frac{1}{4} - 2 - 0 - \frac{1}{4} + 2 + 0 = 0.$$

$$\iint_{A_4} \mathrm{rot}\, F(x,y)\,dx\,dy = \int_1^{\sqrt{6}}\int_{x^2-4}^{2} -x\,dy\,dx = \int_1^{\sqrt{6}} \left[-xy\right]_{y=x^2-4}^{y=2} dx$$

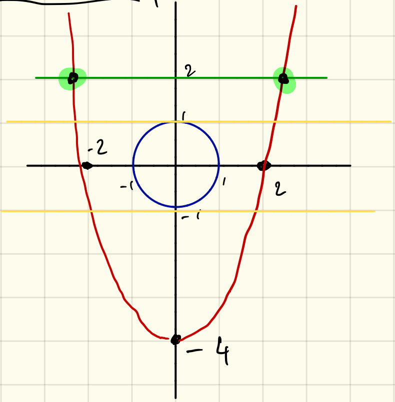
$$= \int_1^{\sqrt{6}} x^3 - 6x\,dx = \left[\frac{1}{4}x^4 - 3x^2\right]_{x=1}^{x=\sqrt{6}}$$

$$= \frac{36}{4} - 18 - \frac{1}{4} + 3 = -9 - \frac{1}{4}.$$

Conclusion :

$$\iint_A \mathrm{rot}\, F(x,y)\,dx\,dy = 9 + \frac{1}{4} + 0 + 0 - 9 - \frac{1}{4} = 0$$ & C'est long!

Remarque : On aurait aussi pu séparer



A en 4 domaines x simples :

$A_1 = \{(x,y) \in \mathbb{R}^2 : y \in [-4,-1], -\sqrt{4+y} \leq x \leq \sqrt{4+y}\}$

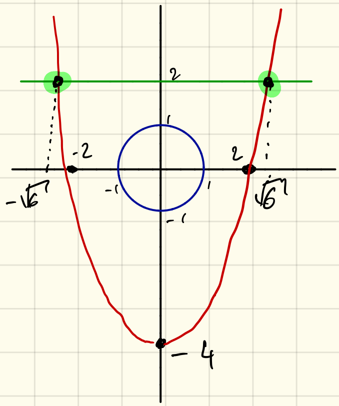
$A_2 = \{(x,y) \in \mathbb{R}^2 : y \in [-1,1],$
$-\sqrt{4+y} \leq x \leq -\sqrt{1-y^2}$

$A_3 = \{(x,y) \in \mathbb{R}^2 : y \in [-1,1],$
$\sqrt{1-y^2} \leq x \leq \sqrt{4+y}\}$

$A_4 = \{(x,y) \in \mathbb{R}^2 : y \in [1,2], -\sqrt{4+y} \leq x \leq \sqrt{4+y}\}$

On a alors, $\iint_{A_0} - dx\,dy = \iint_{A_4} - dx\,dy = 0$, $\iint_{A_1} - dx\,dy = -\int_{A_2} - dx\,dy = \frac{10}{3}$

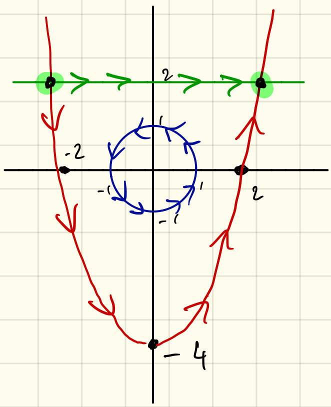
∂A : Le bord de A se sépare naturellement en 3 parties :



$\gamma_1(t) = (t, 2) \quad t \in [-\sqrt{6}, \sqrt{6}]$

$\gamma_2(t) = (t, t^2 - 4) \quad t \in [-\sqrt{6}, \sqrt{6}]$

$\gamma_3(t) = (\cos t, \sin t), \quad t \in [0, 2\pi]$.



γ_1 est orienté négativement

γ_2 est orienté positivement

γ_3 est orienté négativement.

Rem : Dans le livre, γ_1 est paramétré dans l'orientation inverse.